Introduction to ANOVA

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Contents

Introduction

What is ANOVA

- ANOVA is a set of statistical methods used mainly to compare the means of two or more samples. Estimates of variance are the key intermediate statistics calculated, hence the reference to variance in the title ANOVA.
- The different types of ANOVA reflect the different experimental designs and situations for which they have been developed.

Motivation

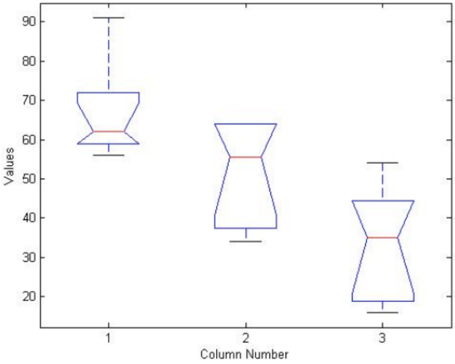
A new drug designed to enhance cognitive performance is ready for testing in animals. Suppose we have a group of young and a group of old rats for the test sets. Each group contains 12 animals and each group is divided into three subgroups, A, B and C. For each group, subgroup A is a control group, Subgroup B receives dose level one of the new drug and subgroup C receives dose level two. Dose level two is twice the dose of dose level one in mg/kg. The average execution time in minutes of a previously well-learned binary choice task for each animal in each group is measured over 3 repetitions.

Motivation

		Drug Level		
		Control	Dose One	Dose Two
Age	Young	56, 62, 57, 72	64, 34, 64, 41	33, 37, 40, 16
	Old	62, 72, 61, 91	64, 48, 34, 63	17, 21, 49, 54

Is there a dose dependent effect of the drug on performance? Is the performance effect different for different age groups?

Motivation



. **Box plots of the performance data by drug dose level.**

There does seem to be an apparent increase in performance (decrease in execution time) as a function of drug level.

Introduction

ANOVA is used to test the following hypotheses (k groups):

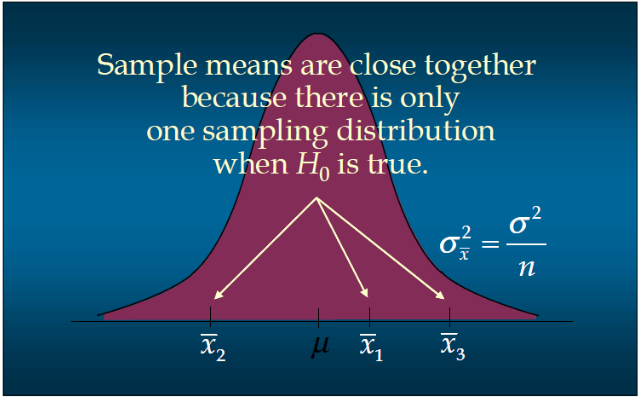
$$H_0 : \mu_1 = \mu_2 = ... = \mu_k.$$

H_a: Not all population means are equal.

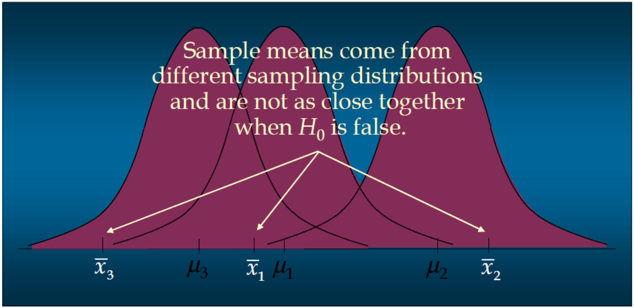
- If H_0 is rejected, we conclude that not all population means are same.
- Rejecting H_0 does not mean that all population means are different. It just means that at least two population means have different values.

Introduction

Sampling Distribution of $\bar{x}$ Given H_0 is True.



Sampling Distribution of $\bar{x}$ Given H_0 is False.



ANOVA and regression

If we assume that there is no age effect. We can then view the problem as a special case of the simple regression model in which the regressor or covariate has three levels: control, dose level one and dose level two. Can the variance in the performance data be explained by taking account of drug level in the analysis? This is an analysis of variance (ANOVA).

ANOVA and regression

In the regression problem, we studied how the variance in the data could be explained by the regressors and we summarized the results in an ANOVA table. Now, in ANOVA, we consider the case in which the regressors have discrete levels.

Assumptions & Differences

ANOVA has the same assumptions as for the regression model:

- Each population is normal and the variances σ^2 are identical.
- The observations are assumed to be independent with each other.

The major difference between ANOVA and regression model is:

- ANOVA cannot model the relationship between IVs and DV, but can look for differences among the groups.

One-Way ANOVA

Notations

n_i: Number of observations in the ith factor level, $i = 1, 2..., I$.
n_T: Total number of observations.
$\bar{Y}_{i\cdot}$: Sample mean for factor level i.

$$\bar{Y}_{i\cdot} = \frac{1}{n_i} \sum_j Y_{ij} \tag{1}$$

$\bar{Y}_{\cdot\cdot}$: Overall or grand mean.

$$\bar{Y}_{\cdot\cdot} = \frac{1}{n_T} \sum_i \sum_j Y_{ij} \tag{2}$$

s_i^2: Sample variance for factor level i.

$$s_i^2 = \frac{1}{n_i - 1} \sum_j (Y_{ij} - \bar{Y}_{i\cdot})^2 \tag{3}$$

One-Way ANOVA

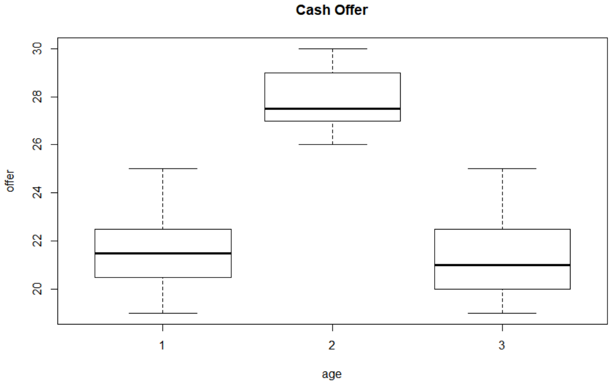
Cash Offers Example:

- Experiment: SAME car was taken by 36 different people (12 young, 12 middle-aged, and 12 elderly) to 36 different dealerships for an offer.
- Goal: Determine if age of owner affects the cash offer made by a dealer for a used car.

One-Way ANOVA

Age	ID	Offer	Age	ID	Offer	Age	ID	Offer
1	1	23	2	1	28	3	1	23
1	2	25	2	2	27	3	2	20
1	3	21	2	3	27	3	3	25
1	4	22	2	4	29	3	4	21
1	5	21	2	5	26	3	5	22
1	6	22	2	6	29	3	6	23
1	7	20	2	7	27	3	7	21
1	8	23	2	8	30	3	8	20
1	9	19	2	9	28	3	9	19
1	10	22	2	10	27	3	10	20
1	11	19	2	11	26	3	11	22
1	12	21	2	12	29	3	12	21

One-Way ANOVA



One-Way ANOVA

Cell Means Model:

$$Y_{ij} = \mu_i + \epsilon_{ij} \tag{4}$$

- μ_i is the (unknown) theoretical mean for all of the observations at level i.
- ϵ_{ij} are independent normal errors(unobservable) with means 0 and variances σ^2.

One-Way ANOVA

Estimates:

- Each group mean is estimated by the mean of the observations within that group:

$$\hat{\mu}_i = \bar{Y}_{i\cdot} = \frac{1}{n_i}\sum_j Y_{ij} \tag{5}$$

- The variance of each group is estimated by:

$$s_i^2 = \frac{1}{n_i - 1}\sum_j (Y_{ij} - \bar{Y}_{i\cdot})^2 \tag{6}$$

One-Way ANOVA

Within-Samples Estimate of Population Variance:
Under the assumption that the variances for all factor levels are the same with the population variance σ^2, we can pool the variances to get an overall variance estimation, which is called the Mean Square Error, and denoted by MSE.

$$MSE = \frac{\sum_i (n_i - 1)s_i^2}{\sum_i (n_i - 1)} = \frac{\sum_i \sum_j (Y_{ij} - \bar{Y}_{i\cdot})^2}{n_T - I} \quad (7)$$

Pooling is weighted according to the number of observations in each factor level.

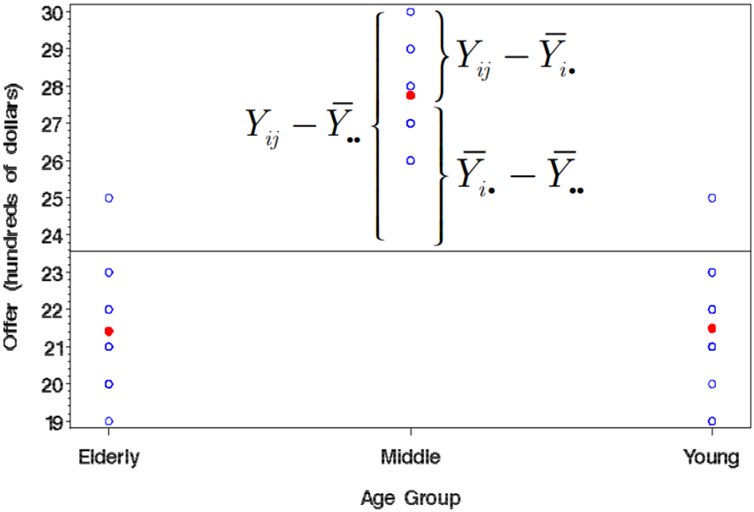
One-Way ANOVA

Partitioning Variation: Break down difference between observation and grand mean into two parts:

$$Y_{ij} - \bar{Y}_{..} = (\bar{Y}_{i.} - \bar{Y}_{..}) + (Y_{ij} - \bar{Y}_{i.}) \tag{8}$$

- $Y_{ij} - \bar{Y}_{..}$: Total deviation.
- $\bar{Y}_{i.} - \bar{Y}_{..}$: Deviation of estimated factor level mean around ground mean. (BETWEEN)
- $Y_{ij} - \bar{Y}_{i.}$: Deviation around estimated factor level mean.(WITHIN)

One-Way ANOVA



One-Way ANOVA

Sums of Squares:
Square both sides, and the cross-terms in $(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}) * (Y_{ij} - \bar{Y}_{i\cdot})$ will cancel.

$$\sum_{i,j}(Y_{ij} - \bar{Y}_{\cdot\cdot})^2 = \sum_{i,j}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 + \sum_{i,j}(Y_{ij} - \bar{Y}_{i\cdot})^2 \tag{9}$$

- $\sum_{i,j}(Y_{ij} - \bar{Y}_{\cdot\cdot})^2$: Sum of squares total.(SSTO)
- $\sum_{i,j}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2$: Sum of squares treatments.(SSTR)
- $\sum_{i,j}(Y_{ij} - \bar{Y}_{i\cdot})^2$: Sum of squares error.(SSE)

One-Way ANOVA

ANOVA(Analysis of Variance) Table

Source	DF	SS	MS
Model/Trt	$I-1$	$\sum_{i,j}(\bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot})^2$	$\frac{SSTrt}{df_{Trt}}$
Error	$n_T - I$	$\sum_{i,j}(Y_{ij}-\bar{Y}_{i\cdot})^2$	$\frac{SSE}{df_{Error}}$
Total	$n_T - 1$	$\sum_{i,j}(Y_{ij}-\bar{Y}_{\cdot\cdot})^2$	

- MODEL line represents variation BETWEEN groups.
- ERROR line represents variation WITHIN groups.

One-Way ANOVA

- $E(MSTr) = \sigma^2 + \frac{1}{I-1}\sum_i n_i(\mu_i - \mu.)^2$, where $\mu.$ is the grand mean.
- $MSTr$ only estimates σ^2 if the population means are equal. If population means are not equal, $MSTr$ estimates a quantity larger than σ^2.
- $E(MSE) = \sigma^2$.

One-Way ANOVA

- If the null hypothesis is true and the ANOVA assumptions are valid, the sampling distribution of $MSTr/MSE$ is an F distribution with $MSTr$ d.f. equal to $k-1$ and MSE d.f. equal to $n_T - k$.
- Ratio $MSTr/MSE$ will be 1 if there is no treatment effect(the means of the k populations are equal) and will be bigger than 1(inflated) if there is a treatment effect, because $MSTr$ overestimates σ^2.
- Hence, we will reject H_0 if the resulting value of $MSTr/MSE$ appears to be too large to have been selected at random from the appropriate F distribution.

One-Way ANOVA

Test for the Equality of k Population Means:

- Hypotheses:

$$H_0: \mu_1 = \mu_2 = ... = \mu_k.$$

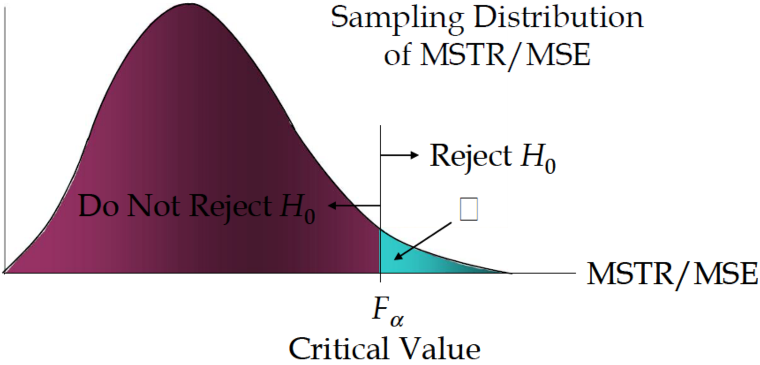
$$H_a\text{: Not all population means are equal.}$$

- Test Statistic:

$$F = \frac{MSTr}{MSE}$$

- Rejection Rule:
  Under H_0, $F \sim F(k-1, n_T - k)$
  - Critical Value Approach: Reject H_0 if $F > F_{critical}$.
  - p-value Approach: Reject H_0 if p-value $< \alpha$

Rejection Region:



One-Way ANOVA

Conclusion:

- ANOVA can be viewed as the process of partitioning the total sum of squares and the degrees of freedom into their corresponding sources: treatments and error.
- Dividing the sum of squares by the appropriate degrees of freedom provides the variance estimates and the F value used to test the hypothesis of equal population means.

Example (Cash Offers):
Data is available on GradQuant Website as "cash.txt".
1. Develop the hypotheses:

$$H_0 : \mu_1 = \mu_2 = \mu_3.$$

H_a: Not all population means are equal.

Note: $\mu_i, i = 1, 2, 3$ is cash offer for the owners in three different age groups.

Level of age	N	---------offer--------- Mean	Std Dev
Elderly	12	21.4167	1.67649
Middle	12	27.7500	1.28806
Young	12	21.5000	1.73205

Example (Cash Offers):

2. Specify the level of significance: $\alpha = 0.05$
3. Compute the value of the test statistic.

- Mean Square Due to Treatments:
  $\bar{Y}_{..} = (21.42 + 27.75 + 21.5)/3 = 23.56$
  $SSTr =$
  $12*(21.42-23.56)^2+12*(27.75-23.56)^2+12*(21.5-23.56)^2 = 316.6$
  $MSTr = 316.55/2 = 158.3$

- Mean Square Due to Error:
  $SSE = 11 * 1.68^2 + 11 * 1.29^2 + 11 * 1.73^2 = 82.27$
  $MSE = 82.3/(36 - 3) = 2.49$
  $F = MSTr/MSE = 158.3/2.49 = 63.6$

Example (Cash Offers):

Source	**DF**	**SS**	**MS**	**F-value**	**p-value**
Model	2	316.7	158.4	63.60	< 0.0001
Error	33	82.2	2.49		
Total	35	398.9			

One-Way ANOVA

- Critical Value Approach:
  With 2 numerator d.f. and 33 denominator d.f., the critical value of F distribution with confidence level $\alpha = 0.05$ is 3.28. So, $F > F_{critical}$, and then the null hypotheses should by rejected.

- *p*-value Approach:
  $p\text{-value} = P(F > F_{test}) = 4.77 * 10^{-12} < 0.0001 < 0.05$

- Conclusion:
  We have sufficient evidence to conclude that the mean number of cash offers for the same car is not the same at all 3 age groups.

Question?

Diagnostics & Remedial Measures

Diagnostics

Assumptions checking: Residuals.

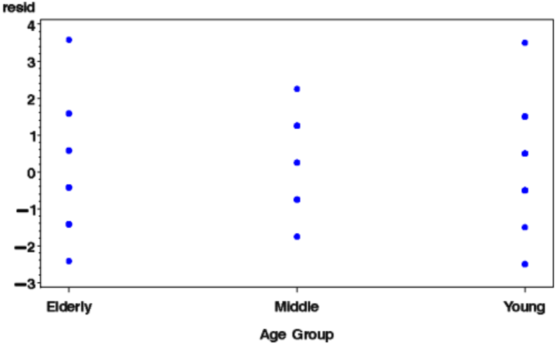
- Predicted values are cell means, $\hat{Y}_{ij} = \bar{Y}_{i.}$
- Residuals are differences between observed values and cell means: $e_{ij} = Y_{ij} - \hat{Y}_{ij} = Y_{ij} - \bar{Y}_{i.}$
- Residual Plot:
  - Plot against fitted values (cell means) or factor levels (check constant variance).
  - Sequence Plot (check independence, when sequence is available/reasonable).
  - Normal Probability Plot (check normality).

Diagnostics

	Age	ID	Offer	Age	ID	Offer	Age	ID	Offer
	1	1	23(1.58)	2	1	28(0.25)	3	1	23(1.5)
	1	2	25(3.58)	2	2	27(-0.75)	3	2	20(-1.5)
	1	3	21(-0.42)	2	3	27(-0.75)	3	3	25(3.5)
	1	4	22(0.58)	2	4	29(1.25)	3	4	21(-0.5)
	1	5	21(-0.42)	2	5	26(-1.75)	3	5	22(0.5)
	1	6	22(0.58)	2	6	29(1.25)	3	6	23(1.5)
	1	7	20(-1.42)	2	7	27(-0.75)	3	7	21(-0.5)
	1	8	23(1.58)	2	8	30(2.25)	3	8	20(-1.5)
	1	9	19(-2.42)	2	9	28(0.25)	3	9	19(-2.5)
	1	10	22(0.58)	2	10	27(-0.75)	3	10	20(-1.5)
	1	11	19(-2.42)	2	11	26(-1.75)	3	11	22(0.5)
	1	12	21(-0.42)	2	12	29(1.25)	3	12	21(-0.5)
s.m			21.42			27.75			21.5

Diagnostics

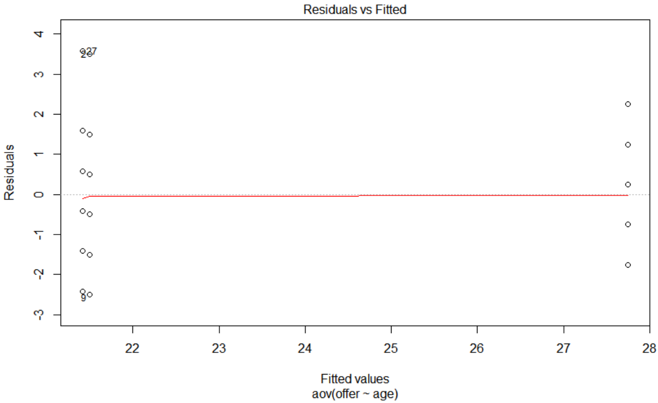
- 1. Constant variance of errors: Residual plot.



Check: large differences in vertical spreads.
No obvious problems with the variance.

Diagnostics

- 1. Constant variance of errors: Residual plot.



Check: large differences in vertical spreads.
No obvious problems with the variance.

Diagnostics

- If residual plots indicate potential problems, can use statistical tests to check.

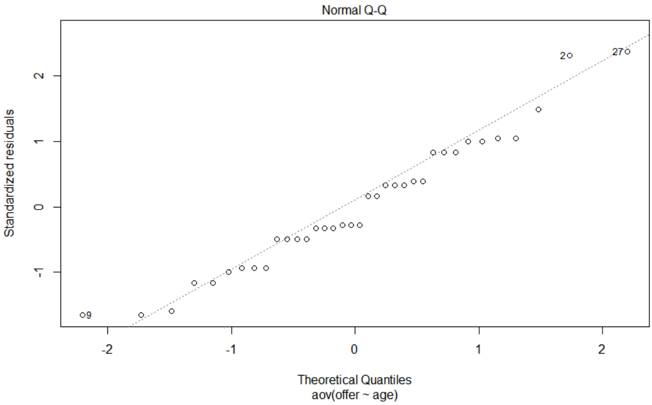
$$H_0 : \sigma_1 = \sigma_2 = ... = \sigma_k.$$
$$H_a : \sigma_i = \sigma_{i'} \text{ for some } i$$

- Brown-Forsythe test.
- Hartley test: simpler test, but requires equal sample sizes and is quite sensitive to departures from normality.
- Levene test: commonly used test, similar to Brown Forsythe.

Note: ANOVA F-test only slightly affected by nonconstant variance as long as sample sizes are equal.

Diagnostics

- 2. Normality assumption: Q-Q plot



No major violations of normality.

Remedial Measure

Transformation:

- If variance proportional to μ_i, then try $\sqrt{Y}$. (sometimes occurs if Y is a count)
- If standard deviation proportional to μ_i, then try $\log Y$.
- If variance proportional to μ_i^2, then try $\frac{1}{Y}$.
- If response is a proportion, try arcsine transformation $Y' = 2 arcsine \sqrt{Y}$.
- Box-Cox:

$$y(\lambda) = \begin{cases} \frac{y^\lambda - 1}{\lambda}, \lambda \neq 0 \\ log(y), \lambda = 0 \end{cases}$$

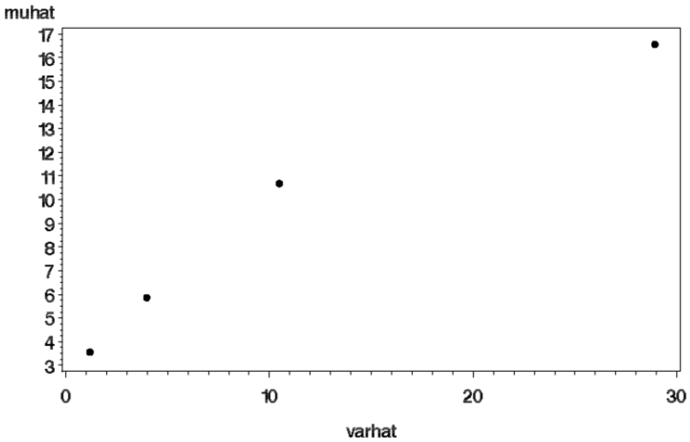
Remedial Measure

Transformation:

- To check whether one of these is applicable, calculate sample factor level variances (s_i^2) and means ($\bar{Y}_{i\cdot}$)
- Create plots: $\bar{Y}_{i\cdot}$ VS s_i^2, $\bar{Y}_{i\cdot}$ VS s_i, and $\bar{Y}_{i\cdot}{}^2$ VS s_i
- If any of the previously mentioned trends appear, use the corresponding transformation.

Remedial Measure

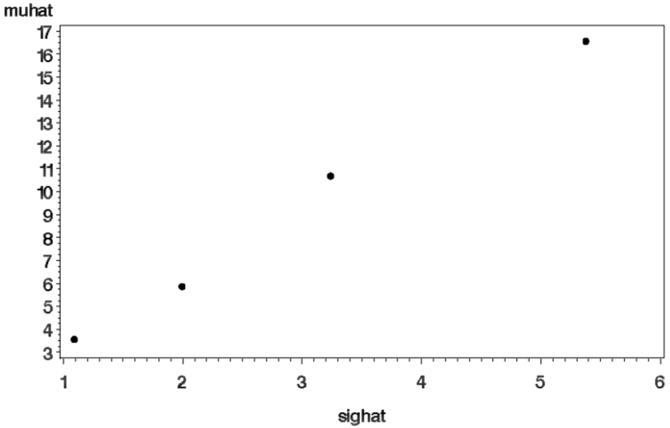
Plot of factor level means vs factor level variances:



Linear relationship suggests Sqrt transformation.

Remedial Measure

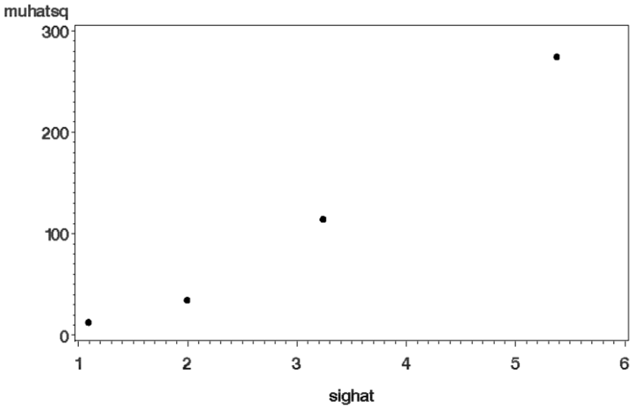
Plot of factor level means vs factor level standard deviation:



Linear relationship suggests Log transformation.

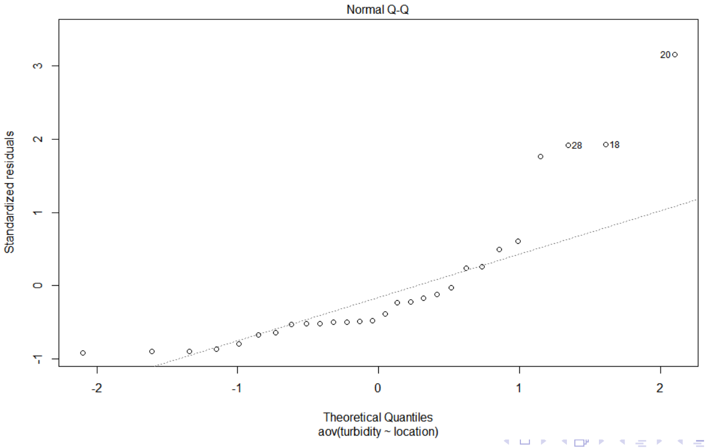
Remedial Measure

Plot of factor level squared means vs factor level standard deviation:

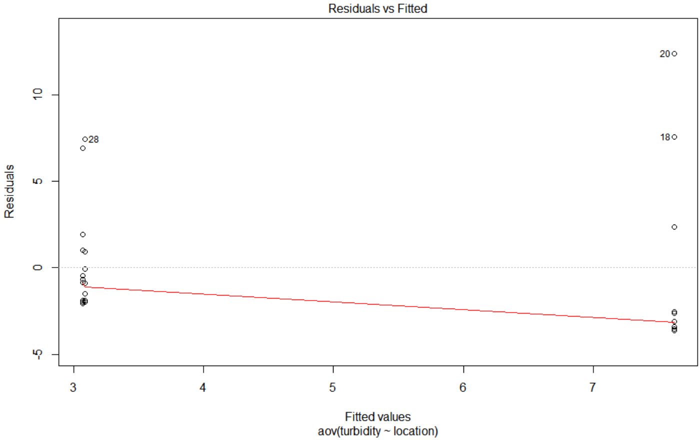


Linear relationship suggests Inverse transformation.

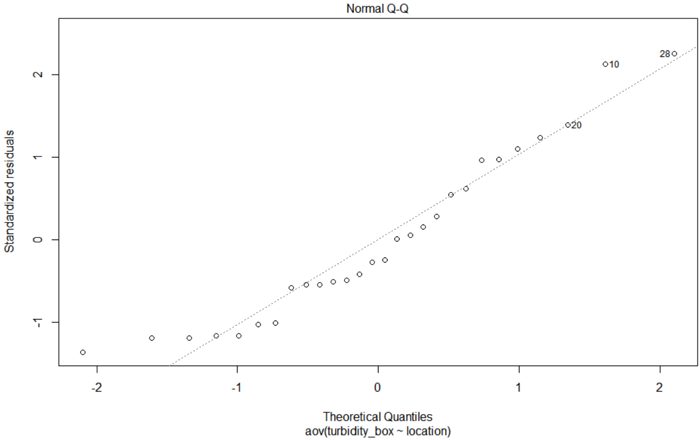
Box-Cox Transformation Example:
Data is available on GradQuant Website as "boxcox.txt".
Before transformation:



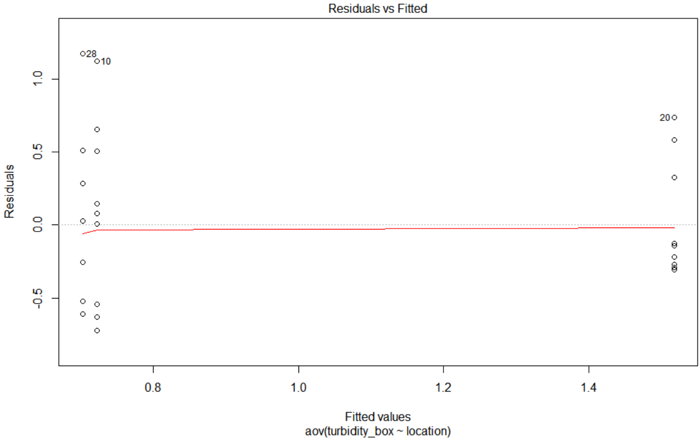
Box-Cox Transformation Example:
Before transformation:



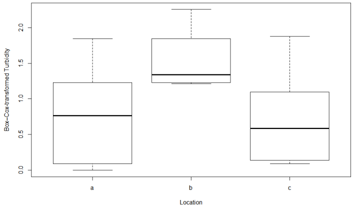
Box-Cox Transformation Example:
After transformation:



Box-Cox Transformation Example:
After transformation:



Box-Cox Transformation Example:

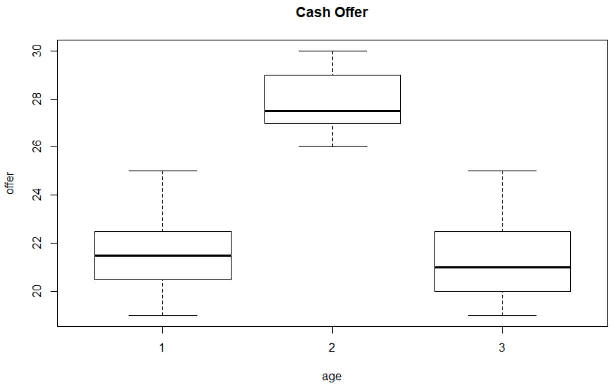


Source	DF	SS	MS	F	p-value
Method	2	4.164	2.0821	6.693	0.0047
Error	25	7.777	0.3111		

Question?

Multiple Comparison

Multiple Comparison



There exist differences among the means, but which means are different?

Multiple Comparison

REVIEW:

- Overall or grand mean: $\bar{Y}_{..} = \frac{1}{n_T} \sum_i \sum_j Y_{ij}$
- Mean for factor level i: $\hat{\mu}_i = \bar{Y}_{i.} = \frac{1}{n_i} \sum_j Y_{ij}$
- Factor Effects estimation: $\hat{\tau}_i = \bar{Y}_{i.} - \bar{Y}_{..}$

Multiple Comparison

REVIEW:

- Under the assumption that Y_{ij}'s are independent and $Var(Y_{ij}) = \sigma^2$

$$Var(\bar{Y}_{..}) = Var(\frac{1}{n_T}\sum_i \sum_j Y_{ij}) = \frac{1}{n_T}\sigma^2$$

- For cell means (fixed level i):

$$Var(\bar{Y}_{i.}) = Var(\frac{1}{n_i}\sum_j Y_{ij}) = \frac{1}{n_i}\sigma^2$$

- For the estimation of above variances, plug in $\hat{\sigma^2} = MSE$
- Standard Error for grand mean: $\hat{SE}(\bar{Y}_{..}) = \sqrt{MSE/N_T}$
- Standard Error for factor level mean: $\hat{SE}(\bar{Y}_{i.}) = \sqrt{MSE/N_i}$

Multiple Comparison

REVIEW:

- Pairwise Comparison(difference between two means):

$$D = \mu_i - \mu_{i'}$$

- Estimation:

$$\hat{D} = \hat{\mu}_i - \hat{\mu}_{i'}$$

- Under the assumption that Y_{ij}'s are independent, $\bar{Y}_{i\cdot}$ and $\bar{Y}_{i'\cdot}$ are independent.

$$Var(\hat{D}) = \frac{\sigma^2}{n_i} - \frac{\sigma^2}{n_{i'}}$$

- The standard error for the difference:

$$\hat{SE}(\hat{D}) = \sqrt{MSE(\frac{1}{n_i} + \frac{1}{n_{i'}})}$$

Multiple Comparison

REVIEW:

- Y_{ij}'s are normally distributed, and $\hat{D}$ is a linear combination of the Y_{ij}'s, so $\hat{D}$ is also normally distributed, Then:

$$\frac{\hat{D}-D}{SE(\hat{D})} \sim t(df = n_t - k)$$

- Then the t-test for pairwise comparison can be formulated as below:

$$H_0 : \mu_i = \mu_{i'},\ H_a : \mu_i \neq \mu_{i'}.$$
$$H_0 : \mu_i - \mu_{i'} = 0,\ H_a : \mu_i - \mu_{i'} \neq 0.$$

- Test Statistics: $\hat{D}/\hat{SE}(\hat{D})$
- Confidence Interval: $\hat{D} \pm t_{critical}\hat{SE}(\hat{D})$

Multiple Comparison

- If we are only interested in doing one test, then we have no problem with the comparison-wise Type I error rate ϵ, which is level of significance associated with a single pairwise comparison.

- If we are interested (at least) in looking at ALL pairwise comparisons, we have issues with experiment-wise Type I error rate ϵ_{EW}, which is the probability of making a Type I error on at least one of the $k(k-1)/2$ pairwise comparisons.

$$\epsilon_{EW} = 1 - (1-\epsilon)^{k(k-1)/2}$$

- The experimentwise Type I error rate gets larger for problems with more populations (larger k).

Multiple Comparison

Least Significant Differences (LSD)

- $LSD = t_{1-\alpha/2}(n_T - k)\sqrt{MSE(\frac{1}{n_i} + \frac{1}{n_{i'}})}$
- If the F-test indicates that not all the population means are equal, then any pair of means that differ by at least LSD are considered to be different.

Multiple Comparison

Tukey Procedure

- Tukey is a method Based on the studentized range distribution, which specifies an EXACT family significance level for comparing all pairs of population means.

- The actual critical value is $q_{critical}/\sqrt{2}$, so the minimum significant difference is:

$$\frac{q_{1-\alpha}(k, n_T-k)}{\sqrt{2}}\sqrt{MSE(\frac{1}{n_i}+\frac{1}{n_{i'}})}$$

- The (1 - α)% confidence interval for the test:

$$H_0 : D = 0, H_A : D \neq 0$$

is:

$$(\bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}) \pm \frac{q_{1-\alpha}(k, n_T-k)}{\sqrt{2}}\sqrt{MSE(\frac{1}{n_i}+\frac{1}{n_{i'}})}$$

Multiple Comparison

Bonferroni Procedure

- Basic idea: Divide alpha by the number of tests in the pairwise comparison.

- The minimum significant difference is:

$$t_{1-\alpha/2g}(n_T - k)\sqrt{MSE(\frac{1}{n_i} + \frac{1}{n_{i'}})}$$

  where g is the total number of tests.

- The (1 - α)% confidence interval for the test:

$$(\bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}) \pm t_{1-\alpha/2g}(n_T - k)\sqrt{MSE(\frac{1}{n_i} + \frac{1}{n_{i'}})}$$

Scheffe Comparisons

- The minimum significant difference is:

$$\sqrt{(k-1)F_{1-\alpha}(k-1, n_T-k)}\sqrt{MSE(\frac{1}{n_i}+\frac{1}{n_{i'}})}$$

- The (1 - α)% confidence interval for the test:

$$(\bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}) \pm \sqrt{(k-1)F_{1-\alpha}(k-1, n_T-k)}\sqrt{MSE(\frac{1}{n_i}+\frac{1}{n_{i'}})}$$

Multiple Comparison

Cash Offers Example

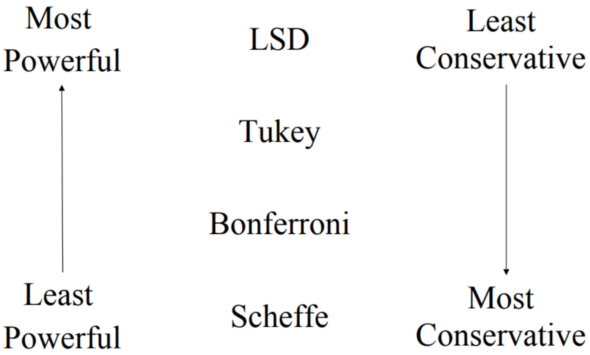
Comparison Type	Critical Value	Minimum Significant Difference
LSD	t = 2.03	1.31
Tukey	$q/\sqrt{2} = 2.45$	1.58
Bonferroni	t = 2.52	1.62
Scheffe	f = 3.28	1.65

Multiple Comparison

- Bonferroni Procedure is used when only interested in a small number of planned pairwise comparisons.
- Tukey Procedure is used when only interested in all (or most) pairwise comparisons of means.
- Scheffe Comparison is used when doing anything that could be considered data snooping - i.e. for any unplanned pairwise comparison.

Multiple Comparison

Significance Level & Power



Multiple Comparison

Other Comparison Procedures

- Duncans Multiple Range Test:
  Used for pairwise comparisons; similar to TUKEY
- Dunnetts test:
  Used for comparing treatments VS control, so (k 1) tests in total.

Question?

Two-Way ANOVA

Two-Way ANOVA

When we have two categorical explanatory variables (Factor A and Factor B):

- Continuous response variables: Y_{ijk}
  Factor A has a levels: $i = 1, ..., a$
  Factor B has b levels: $j = 1, ..., b$
  Observations in cell (i,j) are indexed by k, where $k = 1, ..., n_{ij}$(if balanced design, $n_{ij} \equiv n$).

Cash Offers Example

- Add GENDER as the second factor.
- a = 3 levels of age (young, middle, elderly)
- b = 2 levels of gender (female, male)
- n = 6 observations per age*gender combination (total 36 observations)

Two-Way ANOVA

Age	Gender	ID	Offer	Age	Gender	ID	Offer	Age	Gender	ID	Offer
1	1	1	21	2	1	1	30	3	1	1	25
1	1	2	23	2	1	2	29	3	1	2	22
1	1	3	19	2	1	3	26	3	1	3	23
1	1	4	22	2	1	4	28	3	1	4	21
1	1	5	22	2	1	5	27	3	1	5	22
1	1	6	23	2	1	6	27	3	1	6	21
1	2	1	21	2	2	1	26	3	2	1	23
1	2	2	22	2	2	2	29	3	2	2	19
1	2	3	20	2	2	3	27	3	2	3	20
1	2	4	21	2	2	4	28	3	2	4	21
1	2	5	19	2	2	5	27	3	2	5	20
1	2	6	25	2	2	6	29	3	2	6	20

Two-Way ANOVA

Factor Effects Model:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

where $\epsilon_{ijk} \sim N(0, \sigma^2)$ and are independent.

- Grand mean:
  μ is estimated by $\hat{\mu} = \bar{Y}_{...}$
- Main Effects:
  α_i can be estimated by $\hat{\alpha}_i = \bar{Y}_{i..} - \bar{Y}_{...}$
  β_j can be estimated by $\hat{\beta}_j = \bar{Y}_{.j.} - \bar{Y}_{...}$
- Interactions:
  $(\alpha\beta)_{ij}$ can be estimated by $(\hat{\alpha\beta})_{ij} = \bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}$

Summary of Estimates

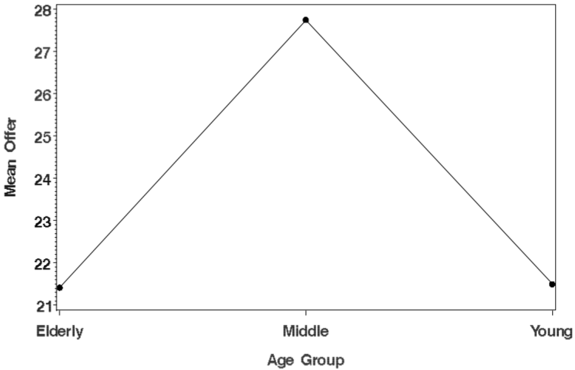
$$\hat{\alpha}_{male} = 0.3889$$
$$\hat{\alpha}_{female} = -0.3889$$

$$\hat{\beta}_{eld} = -2.14$$
$$\hat{\beta}_{mid} = 4.58$$
$$\hat{\beta}_{yng} = -2.44$$

$$\left(\widehat{\alpha\beta}\right)_{m,e} = 0.53 \qquad \left(\widehat{\alpha\beta}\right)_{f,e} = -0.53$$
$$\left(\widehat{\alpha\beta}\right)_{m,m} = -0.31 \qquad \left(\widehat{\alpha\beta}\right)_{f,m} = 0.31$$
$$\left(\widehat{\alpha\beta}\right)_{m,y} = -0.22 \qquad \left(\widehat{\alpha\beta}\right)_{f,y} = 0.22$$

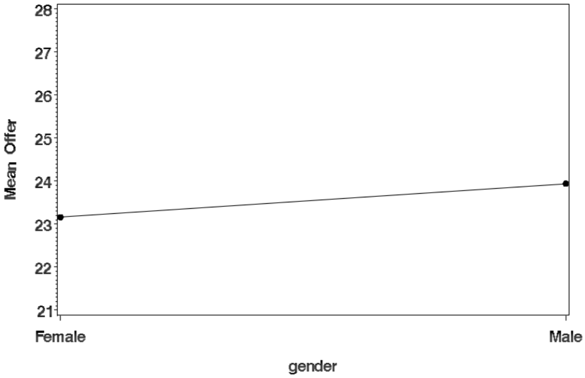
Two-Way ANOVA

Mean Effects(Age)



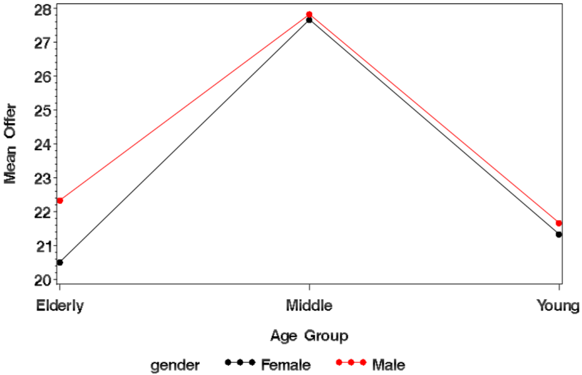
Two-Way ANOVA

Mean Effects(Gender)



Two-Way ANOVA

Interaction Effect



Two-Way ANOVA

ANOVA table

- Sum squares(SS) of model in One-way ANOVA, is partitioned into SSA, SSB, and SSAB in Two-way ANOVA.
- The corresponding degrees of freedom are $a-1$, $b-1$, and $(a-1)(b-1)$.
- Sum squares of Error is derived by subtracting the above from Sum squares Total.

F-tests

- Main Effect of Factor A:

$$H_0 : \alpha_1 = \alpha_2 = ... = \alpha_a = 0$$
$$H_a\text{: Not all } \alpha_i \text{ equal 0.}$$

- Main Effect of Factor B:

$$H_0 : \beta_1 = \beta_2 = ... = \beta_b = 0$$
$$H_a\text{: Not all } \beta_j \text{ equal 0.}$$

- Interaction Effect:

$$H_0\text{: all } (\alpha\beta)_{ij} \text{ equal 0.}$$
$$H_a\text{: Not all } (\alpha\beta)_{ij} \text{ equal 0.}$$

Two-Way ANOVA

ANOVA Results

Source	DF	SS	MS	F-value	p-value
Age	2	316.72	158.36	66.29	< 0.0001
Gender	1	5.44	5.44	2.28	0.1416
Age*Gender	2	5.06	2.53	1.06	0.3597
Error	30	71.67	2.39		
Total	35	398.89			

Multiple ANOVA analysis

Multiple ANOVA analysis

General Strategy

- 1. Set up model with main effects and interaction(s), check assumptions, and examine interaction(s).
- 2. If no significant interaction, examine main effects individually, using appropriate adjustments for multiple comparisons, main effects plots, etc.
- 3. If interaction is significant, determine whether interactions are important. If not, can examine main effects as in Step 2.

Multiple ANOVA analysis

- 4. If interaction present important, determine whether interaction is simple or complex.
- 5. For simple interactions, can still talk about the main effects of A at each level of B.
- 6. For complex interaction, must simply consider all pairs of levels as separate treatments.

Question?

- For hands on help with your analyses, stop by our drop in hours or sign up for a consultation.
- Welcome to the last workshop this summer!
- If you have any workshop requests, now is the time to ask! We will be setting our fall schedule soon.
- For more details, visit our website:
  GradQuant.ucr.edu

Thank You
Welcome to GradQuant